

New Math – Logarithms Made Easy
A New Approach to Expressing Exponentiation and Logarithms
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It seems to me that the way we go about teaching logarithms is all wrong. Or to put it a little less starkly, I think there is a better way to explain, define, and implement logarithms, roots, and exponents.

To go back to basics, multiplication is just repetitive addition. It is a shortcut, or shorthand, for repeated addition. For example, we compact $3+3+3+3+3$ into 3×5 . To undo the product $3 \times 5 = 15$, division was created. Thus to get back the multiplicand in $3 \times 5 = 15$ we divide by 5, getting 3. Likewise, to get back the multiplier we divide by the multiplicand 3, giving us the multiplier 5

To go back one step further, we do the same in addition. Take the sum $5+6=11$. 5 is the augend and 6 is the addend. To undo addition, subtraction was created. So if we know that the addend is 6, we subtract 6 (from the sum, 11) to find the augend, 5.

Similarly exponentiation is just repeated multiplication. Thus $5 \times 5 \times 5$ becomes 5^3 . (Here the exponent is 3 and the base is 5). Now suppose we have the number 125 (the expansion) and we know that this was created from the exponent 3, but we do not know the base. How do we find it? What do we do to $x^3 = 125$ to find the value of the x? For this, roots were created. Thus, the 3rd root of 125 yields 5, the base.

$$\begin{aligned} \sqrt[3]{x^3} &= \sqrt[3]{125} && \text{I} \\ x &= 5 \end{aligned}$$

Now to undo multiplication, we only need one operation, division. That is because multiplication is commutative. $5 \times 6 = 6 \times 5$. However, exponentiation is not commutative. $5^4 \neq 4^5$. Taking roots will get back the base. But we need another operator to get back the exponent. Thus logs. To undo $5^x = 125$ we apply logs. Thus, log to the base 5 of 125 is 3.

$$\begin{aligned} \log_5 5^x &= \log_5 125 && \text{II} \\ x &= 3 \end{aligned}$$

Thus, logs are the way of undoing an expression “ a^x ” and finding the x.

So, instead of teaching that logs are an aid to multiplying

$$\begin{aligned} \log ab &= \log a + \log b \\ ab &= \text{antilog} (\log a + \log b) \end{aligned}$$

or teaching that they are an area under a hyperbola, approach them simply as the way of undoing exponentiation - finding the exponent!

Thus, logs can be introduced in school at the same time as finding square roots, and roots in general. Logs are really no more complicated than finding roots.

However, the nomenclature we use is cumbersome. Instead, I use the symbol “ \uparrow ” for exponentiation, “ \downarrow ” for the log function, and “ $\sqrt{}$ ” for the root function.

Thus, 5^3 is $5\uparrow 3$
 $\log_5 125$ is $125\downarrow 5$
 $\sqrt[3]{125}$ is $3\sqrt{125}$

Both systems of notation look essentially the same, except that in the case of logs, the base now comes at the end, which is really more logical. *

$$5\uparrow 3 = 125$$

$$125\downarrow 5 = 3$$

$$3\sqrt{125} = 5$$

In mathematical terms, the base (5) raised to the power (3) equals the expansion (125). I like to read this “5 up 3 equals 125”. Likewise, I read $125\downarrow 5$ as “125 down 5” or “125 log 5” and $3\sqrt{125}$ as “3 root 125” or 3rd root 125

This nomenclature has a number of advantages, not the least of which, it is uncluttered, and can all be written on one line, without subscripts and superscripts, certainly an advantage with computers and in programming.

Also, and Most Important, it makes the solving of equations more transparent.

Take $b\uparrow p = s$ (base \uparrow power = expanSion)

To solve for b, the base, we take the p^{th} root of both sides ($p\sqrt{\quad}$) – analagous to I above
 $p\sqrt{(b\uparrow p)} = p\sqrt{s}$ or $b = p\sqrt{s}$ (1)

To solve for p, the exponent or the power, we take the \log_b of both sides ($\downarrow b$) – as in II
 $(b\uparrow p)\downarrow b = s\downarrow b$ or $p = s\downarrow b$ (2)

Now let’s look at the 2 new expressions introduced in (1) and (2) above, $p\sqrt{s}$ and $s\downarrow b$.

Take $s\downarrow b = p$ (expansion \downarrow base = power) or log to base b of the expansion = p

To solve for s, we apply “ $b\uparrow$ ” to both sides of the equation.

$$b\uparrow(s\downarrow b) = b\uparrow p \quad \text{or } s = b\uparrow p \quad (3)$$

To get back the b, we apply “ \sqrt{s} ” to both sides of the equation.

$$(s\downarrow b)\sqrt{s} = p\sqrt{s} \quad \text{or } b = p\sqrt{s} \quad (4)$$

Take $p\sqrt{s} = b$ **

To solve for p, we apply “ $s\downarrow$ ” to both sides of the equation

$$s\downarrow(p\sqrt{s}) = s\downarrow b \quad \text{or } p = s\downarrow b \quad (5)$$

To solve for s, we apply “ $\uparrow p$ ” to both sides of the equation

$$p\sqrt{s}\uparrow p = b\uparrow p \quad \text{or } s = b\uparrow p \quad (6)$$

* And necessary to allow (2) thru (5) above to work, with unknown in the middle.

See Addendum I

** p, when it occurs in front of the $\sqrt{\quad}$ sign, is frequently called the index

To summarize, then, there are 3 operators - \uparrow , \downarrow , and $\sqrt{}$. These lead to 3 possible equations.

- 1) $b \uparrow p = s$
- 2) $s \downarrow b = p$
- 3) $p \sqrt{s} = b$

In each of these 3 operations, we may wish to solve for the 1st term on the left side of the equation, or we may want to solve for the 2nd term on the left side of the equation. This then leads to 6 (possible) solutions. These are:

- 1) $b \uparrow p = s \rightarrow \rightarrow p \sqrt{(b \uparrow p)} = p \sqrt{s}$ or $b = p \sqrt{s}$ (solving for the base)
 $\rightarrow \rightarrow (b \uparrow p) \downarrow b = s \downarrow b$ or $p = s \downarrow b$ (solving for the power)
- 2) $s \downarrow b = p \rightarrow \rightarrow b \uparrow (s \downarrow b) = b \uparrow p$ or $s = b \uparrow p$
 $\rightarrow \rightarrow (s \downarrow b) \sqrt{s} = p \sqrt{s}$ or $b = p \sqrt{s}$
- 3) $p \sqrt{s} = b \rightarrow \rightarrow s \downarrow (p \sqrt{s}) = s \downarrow b$ or $p = s \downarrow b$
 $\rightarrow \rightarrow (p \sqrt{s}) \uparrow p = b \uparrow p$ or $s = b \uparrow p$

The recipe, then, for solving any of the 3 equations, mechanically, is just follow these 2 rules

- (1) Use the operators $\sqrt{}$, \uparrow , \downarrow , and $\sqrt{}$ in just that order. **
- (2) Arrange it so that the symbol you want to solve for is in the middle.

Example: to undo $b \uparrow p = s$

To solve for p

- 1) We want p in the middle. Take the next operator after \uparrow , (ie: \downarrow), and place it after $(b \uparrow p)$
 $(b \uparrow p) \downarrow _ = s \downarrow _$
- 2) Reuse the b after the \downarrow , so that the p is in the middle, between 2 b's
 $(b \uparrow p) \downarrow b = s \downarrow b$ or $p = s \downarrow b$

To solve for b

- 1) We want b in the middle. Take the operator before \uparrow (ie: $\sqrt{}$) and place it before $(b \uparrow p)$
 $_ \sqrt{(b \uparrow p)} = _ \sqrt{s}$
- 2) Place p before the $\sqrt{}$, so that we take the pth root of $(b \uparrow p)$ and of s.
 $p \sqrt{(b \uparrow p)} = p \sqrt{s}$ or $b = p \sqrt{s}$

So now we can readily solve any of the 3 equations

$$\begin{aligned} x \uparrow y &= z \\ x \downarrow y &= z \\ x \sqrt{y} &= z \end{aligned}$$

for either x or y.

** Note that in equations 1) thru 6) on page 2, the operators always occur in the sequence $\sqrt{} \uparrow \downarrow \sqrt{}$ and in 1)to 3) above and 1)to 6) below they always occur in this same sequence, $\sqrt{} \uparrow$, or $\uparrow \downarrow$, or $\downarrow \sqrt{}$

Again to summarize;

| <u>↑↓√ Form .</u> | <u>Conventional Form</u> | <u>*</u> |
|-----------------------------------|---------------------------|----------|
| $(b \uparrow p) \downarrow b = p$ | $\log_b b^p = p$ | (2) |
| $(s \downarrow b) \sqrt{s} = b$ | $\log_b b^s \sqrt{s} = b$ | (4) |
| $(p \sqrt{s}) \uparrow p = s$ | $(p \sqrt{s})^p = s$ | (6) |
| $p \sqrt{(b \uparrow p)} = b$ | $p \sqrt{(b^p)} = b$ | (1) |
| $b \uparrow (s \downarrow b) = s$ | $b^{\log_b s} = s$ | (3) |
| $s \downarrow (p \sqrt{s}) = p$ | $\log_{p \sqrt{s}} s = p$ | (5) |

All of the 6 equations on the left above are associative – that is, they do not need parentheses.

The $\uparrow \downarrow \sqrt{}$ system could be especially helpful and convenient when using the hand held calculator. Mine is a TI-89. It has only an ^ key (equivalent to the “↑” in the $\uparrow \downarrow \sqrt{}$ system), and an “ln” key. To get a log function such as $125 \downarrow 5$, I have to remember that $\log_b a = (\log_c a) / (\log_c b)$ ** and so enter $\ln(125) / \ln(5)$ – (9 steps on the TI-89). If the calculator (or computer) had a \downarrow key, I could enter directly the $125 \downarrow 5$, punch enter, and get 3 (3 steps)

Likewise, if the calculator had a “√” key, I could solve directly $3 \sqrt{125}$ – (3 steps), instead of entering $125^{(1 \div 3)}$ – (7 steps) ***

Slightly more complicated calculations, such as

$$(3 \sqrt{4096}) \downarrow 4 \quad \text{vs.} \quad \ln(4096^{(1 \div 3)}) \div \ln(4) \quad **$$

(7 steps) (15 steps)

would be vastly simpler to enter

* This is the numbering used on page 2 to identify & introduce these same 6 formulas

** Formula 4) in Appendix - $a \downarrow b = (a \downarrow e) / (b \downarrow e)$

*** Formula 1) in Appendix

CONCLUSION: The $\uparrow\downarrow\sqrt{\quad}$ system has a number of advantages.

- 1) Everything is written on one line, without subscripts and superscripts, facilitating programing, and writing papers such as this, and adding to legibility
- 2) It could facilitate entry of calculations into the hand held calculator (as well as the computer)
- 3) It would make the teaching of logarithms, roots, and exponents easier, clearer, and more readily understandable
- 4) The reading of an expression is simplified. Instead of saying, for 5^3 , “5 to the 3rd power”, we can say – “5 up 3”. For $\log_{10} 125$, instead of “log to the base 10 of 125”, say “125 down 10”, and instead of “the 3rd root of 125”, say “3 root 125”
(this is similar to saying “three times five” for “three multiplied by five”)
- 5) It would make the use and implementation of logarithms, etc. by students more logical, and more mechanical. I know the idea of “mechanical” and “by recipe” may bother some mathematicians, but I know this system has made my hand calculations much easier and less prone to mistakes. The system is so logical, and beautifully symmetrical (see the left hand column at the top of page 4).

Most important – all 3 operations, \uparrow , \downarrow , and $\sqrt{\quad}$, have the same basic structure, and we use the same basic procedures for unraveling each of them.

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Addendum I

These solutions depend on the following facts

$$\begin{array}{ll} (a \uparrow b) \downarrow a = b & \text{and} & a \uparrow (b \downarrow a) = b \\ (a \downarrow b) \sqrt{a} = b & & a \downarrow (b \sqrt{a}) = b \\ (a \sqrt{b}) \uparrow a = b & & a \sqrt{(b \uparrow a)} = b \end{array}$$

Note that operators always occur in the order $\downarrow \uparrow$, $\downarrow \sqrt{}$, or $\sqrt{} \uparrow$

Note also that the unknown to be solved for always occurs in the middle, keeping everything symmetrical and easy to apply.

Note additionally that, unlike $\log_b s = p$, this whole system depends on the base occurring after the expansion - $s \downarrow b = p$. Otherwise the system falls down

Thus we can easily and quickly solve any of the 3 equations.

$$x \uparrow y = z \quad (1)$$

$$x \downarrow y = z \quad (2)$$

$$x \sqrt{y} = z \quad (3)$$

for either x or y

For example: To solve (3) for x

$$\text{Take: } x \sqrt{2187} = 3$$

I would have difficulty solving this by conventional means

But to solve by the " $\uparrow \downarrow \sqrt{}$ " method is easy

$$x \sqrt{2187} = 3$$

$$2187 \downarrow x \sqrt{2187} = 2187 \downarrow 3$$

$$x = \ln 2187 \div \ln 3 = 7$$

(if Calculator had a " \downarrow " key, could skip this step)

Addendum I (continued)

OK, to solve $\sqrt[x]{2187}=3$ by conventional means:

$$\begin{aligned}\sqrt[x]{2187} &= 3 \\ 2187^{1/x} &= 3 \\ \ln(2187^{1/x}) &= \ln 3 \\ (1/x) \ln 2187 &= \ln 3 \\ 1/x &= \ln 3 \div \ln 2187 \\ x &= \ln 2187 \div \ln 3 \\ x &= 7\end{aligned}$$

Another example: To solve (2) for y

$$\begin{aligned}\text{Take } 1024 \downarrow y &= 5 \\ 1024 \downarrow y \sqrt[5]{1024} &= 5 \sqrt[5]{1024} \\ y &= 1024 \uparrow (1/5) = 4 \quad *\end{aligned}$$

By conventional means

$$\begin{aligned}\log_y 1024 &= 5 \\ y^{\log_y 1024} &= y^5 \\ 1024 &= y^5 \\ \sqrt[5]{1024} &= \sqrt[5]{y^5} \\ \sqrt[5]{1024} &= y \\ y &= 1024^{(1/5)} = 4\end{aligned}$$

You don't really have to memorize all those formulas in the Appendix *

*And if the calculator had a “ \downarrow ” key and a “ $\sqrt{}$ ” key, you wouldn't have to use any of them.

Addendum I (concluded)

Put another way:

here is a different illustration of the simplification brought about by my system (the $\uparrow\downarrow\sqrt{}$ system). The 3 formulas in the left hand column below, and the 3 formulas in the right hand column, are all completely covered, in my system, by the 3 simple formulas in the center column.

$$\begin{aligned} \log_a a^b &= b \\ \log_b a \sqrt{a} &= b \\ (a\sqrt{b})^a &= b \end{aligned}$$

$$\begin{aligned} a\uparrow b\downarrow a &= b \quad (\text{A}) \\ a\downarrow b\sqrt{a} &= b \quad (\text{B}) \\ a\sqrt{b}\uparrow a &= b \quad (\text{C}) \end{aligned}$$

$$\begin{aligned} a^{\log_a b} &= b \\ \log_a (b\sqrt{a}) &= b \\ a\sqrt{b^a} &= b \end{aligned}$$

Formulas in the middle column are associative, and do not need parentheses. Left hand column is equivalent to placing parens around the first 3 characters in center column (ie: $(a\uparrow b)\downarrow a = b$). Right hand col. is the same as placing parens around the 3rd, 4th & 5th characters (ie: $a\uparrow(b\downarrow a)=b$). However, the center column without parens serves to replace either the left or right hand column.

The point here is that the 3 simple equations in the center column completely replace all 6 of the complicated formulas in the right and left hand columns. To get the 3 simple equations, just space the letters a,b,a between the operators $\uparrow\downarrow$, $\downarrow\sqrt{}$, or $\sqrt{}\uparrow$.

Why is all this significant? The 6 formulas in the 2 outside columns are necessary to solve equations involving logs, roots, and exponents, with 2 knowns and one unknown.

For example:

Take : $\log_x 2187 = 7.$

Solution: $\log_x 2187 \downarrow x = 7$
 $\sqrt[7]{2187} = x$
 $x = 2187^{(1\div 7)}$
 $x = 3$

Or * $x^{\log_x 2187} = x^7$
 $2187 = x^7$
 $2187^{(1\div 7)} = x$
 $x = 3$

But you can readily solve this by my method. Just follow the 2 rules on page 3, as follows:

Problem: $2187\downarrow x = 7$

Solution: $2187\downarrow x \sqrt{2187} = 7\sqrt{2187}$
 $x = 3$

Applying formula (B) above

Simple, straightforward, and (almost) foolproof

*Alternate Solution – This is how I would solve it, conventionally